CLOSED SETS, LIMIT POINTS AND CONTINUOUS FUNCTIONS

In a topological space $(X, \mathcal{T})$, recall that $A \subseteq X$ is closed if $X \setminus A$ is open.

Thm: Given $(X, \mathcal{T})$ as above,
(i) any arbitrary intersection of closed sets is closed
(ii) any finite union of closed sets is closed
(iii) $\emptyset$ and X are closed.

Eg: Singleton sets are always closed.
In any metric space (X, d), sets of the form $\{y \in X : d(x,y) \leq \varepsilon\}$ for some $x \in X$, $\varepsilon > 0$ are closed. The set above is called the closed ε-ball centered at x.

Def: Let $(X, \mathcal{T})$ be a topological space $A \subseteq X$.
The closure of A is the intersection of all closed sets in X containing A.

The interior of A is the union of all open sets in X that are contained in A.

We use the notation $Cl\,A$ and $Int\,A$ for the closure and interior of A resp.

Prop: $Int\,A \subseteq A \subseteq Cl\,A$ for all subsets A of a topological space $(X, \mathcal{T})$.

Proof:

Eg: 1. In $(\mathbb{R}, \mathcal{T}_d)$, $Cl\,(a,b) = [a,b]$ and $Int\,(a,b) = (a,b)$.

Proof:

2. In any metric space (X, d), $Cl\,B_d(x, \varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$ and $Int\,B_d(x,\varepsilon) = B_d(x,\varepsilon)$.

Proof:

Theorem: Let $(X, \mathcal{T})$ be a top. space and $A \subseteq X$.
1. A point $x \in X$ belongs to $Cl\,A$ if and only if for every neighborhood U of x, $U \cap A \neq \emptyset$.
2. Suppose $\mathcal{T}$ is generated by a basis $\mathcal{B}$, then $x \in Cl\,A \iff \forall B \in \mathcal{B}$ containing x, we have $B \cap A \neq \emptyset$.

Proof:

Eg: In $(\mathbb{R}, \mathcal{T}_d)$, if $B = \{1/n : n \in \mathbb{N}\}$, then $Cl\,B = B \cup \{0\}$.

Exercise: Find $Int\,B$ where B is as above.

Def: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. A point $x \in X$ is called a "limit point of A" [alternatively, cluster point or accumulation point] if $\forall$ neighborhood U of x, the intersection $U \cap A$ contains a point $y \neq x$.

Prop: In the setting above, $x \in X$ is a limit point of $A \iff x \in Cl\,A \setminus \{x\}$.

Proof:

Eg: In $(\mathbb{R}, \mathcal{T}_d)$, $A = \{1/n : n \in \mathbb{N}\}$. 0 is the only limit point of A.

Proof:

Prop: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Let A' be the set of limit points of A. Then
$$Cl\,A = A \cup A'.$$

Proof:

Prop: Let (X, d_X), (Y, d_Y) be metric spaces. A function $f : X \longrightarrow Y$ is continuous $\iff$ if for every open set $V \subseteq Y$, the set $U = f^{-1}(V)$ is open.

Proof: $\Rightarrow$: Let $V \subseteq Y$ be open, and let $U = f^{-1}(V)$. Given $x \in U$, $\exists \varepsilon > 0$ s.t. $B_{d_Y}(f(x), \varepsilon) \subset V$. By continuity of f, $\exists \delta > 0$ s.t. $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon)$. In other words, $B_{d_X}(x, \delta) \subset f^{-1}(V) = U$.

$\Leftarrow$ Suppose $V \subseteq Y$ open, $f^{-1}(V)$ is open in (X, d).
Given $x \in X$ and $\varepsilon > 0$, $B_{d_Y}(f(x), \varepsilon)$ is open.
so $U = f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open.
$\because x \in U$, $\exists \delta > 0$ s.t. $B_{d_X}(x, \delta) \subseteq U$.
Clearly, $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \varepsilon)$.

Corollary: If (X, d_X) is a discrete metric space, every function $f : X \longrightarrow Y$ is continuous.

Proof: Every subset of X is open, so in particular $f^{-1}(V)$ is open $\forall V \subseteq Y$ that is open.